Lecture no. 8

Measure and Integration

3/12/10

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$\mathcal{C}$ – an algebra

$\mu : \mathcal{C} \longrightarrow [0, +\infty]$ is f.a.

Suppose $\mu(X) < +\infty$

Then f.a. of $\mu \Rightarrow \mu$ is monotone

Thus $\forall\ A \subseteq X$

$$\mu(A) \leq \mu(X) < +\infty$$

$$\Rightarrow \mu(A) < +\infty \ \forall\ A \subseteq X$$

Converse is true $\because X \in \mathcal{C}$

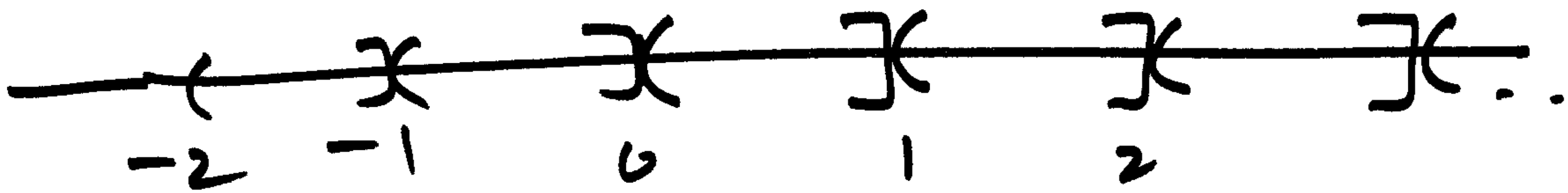
$$\mu(X) < +\infty$$

$$\lambda : \mathcal{I} \longrightarrow [0,+\infty]$$

$$\mathbb{R} = (-\infty,+\infty)$$

$$\lambda(\mathbb{R}) = +\infty$$

$$\mathbb{R} = \bigsqcup_{n\in\mathbb{Z}} (n, n+1]$$



$$\lambda(n,n+1] = 1 < \infty \quad \forall n.$$

λ is σ-finite.
It is not finite

$[a,b]$

$\mathcal{I}_{a,b}$ = All subintervals of $[a,b]$

$$\lambda : \underline{\mathcal{I}_{a,b}} \longrightarrow [0, b-a]$$

$\lambda(I)$ = length of I, $I \subseteq [a,b]$

$$\lambda([a,b]) = b-a < +\infty.$$

λ restricted to subintervals in $[a,b]$ is finite.

(4)

$$A = \bigsqcup_{i=1}^{\infty} A_i.$$

If $A = \phi$, then $A_i = \phi \ \forall i$

$$\Rightarrow \mu(A) = 0 = \sum_{i=1}^{\infty} \mu(A_i)$$

If $A \neq \phi$, $\Rightarrow \exists$ at least one i such that $\underline{A_{i_0} \neq \phi}$

Then $\cancel{\exists A_{i_0} \neq \phi}$

$$\Rightarrow \mu(A) = +\infty = \sum_{i=1}^{\infty} \underline{\mu(A_i)}$$

$(\mu(A_{i_0}) = +\infty)$

(5)

$\mathcal{C}$ — Semi-algebra

$\mathcal{A}(\mathcal{C})$ = Algebra generated by $\mathcal{C}$

<u>Given</u> $\mu_1(E) = \mu_2(E) \ \forall \ E \in \mathcal{C}$

<u>To show</u> $\mu_1(A) = \mu_2(A) \ \forall \ A \in \mathcal{A}(\mathcal{C})$.

Pf. Let $A \in \mathcal{A}(\mathcal{C})$ ✓

$$\Rightarrow \quad A = \bigsqcup_{i=1}^{n} C_i, \quad C_i \in \mathcal{C}$$

Then

$$\underline{\mu_1(A)} = \mu_1\left(\bigsqcup_{i=1}^{n} C_i\right)$$

$$= \sum_{i=1}^{n} \mu_1(C_i) = \sum_{i=1}^{n} \mu_2(C_i)$$

$$= \underline{\mu_2(A)}$$

$\mu_1, \mu_2 : \mathcal{S}(\mathcal{C}) \longrightarrow [0, +\infty]$

measures

$\mathcal{C}$ – semi-algebra

$\mathcal{S}(\mathcal{C})$ = σ-algebra generated by $\mathcal{C}$

<u>Given</u> $\mu_1(A) = \mu_2(A) \quad \forall \ A \in \mathcal{C}$

<u>To show</u> $\mu_1(A) = \mu_2(A) \quad \forall \ A \in \mathcal{S}(\mathcal{C})$.

We may assume $\mathcal{C}$ is an algebra

$\mu_1(A) = \mu_2(A) \ \forall \ A \in \mathcal{A}(\mathcal{C})$

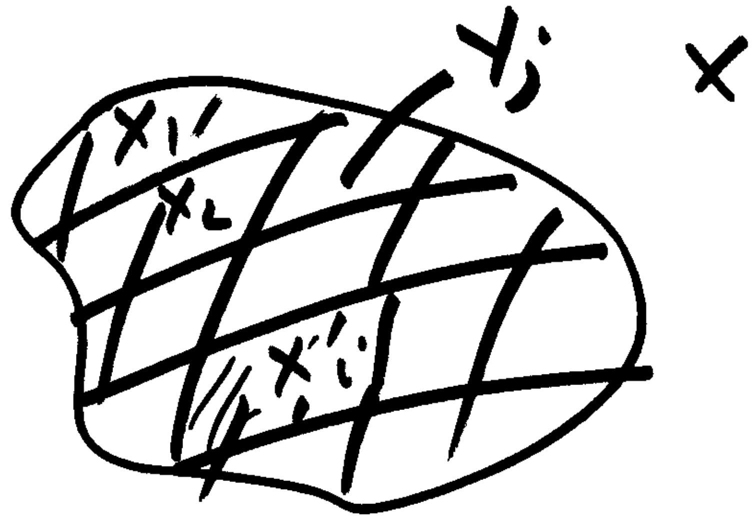
$\Longrightarrow \mu_1(A) = \mu_2(A)$ on $\underline{\mathcal{S}(\mathcal{C})} = \underline{\mathcal{S}(\mathcal{A}(\mathcal{C}))}$

(8)

$$\Rightarrow \quad X = \bigsqcup_{i=1}^{\infty} \bigsqcup_{j=1}^{\infty} (X_i \cap Y_j) \checkmark$$

Note $\mu_1(\underline{X_i \cap Y_j}) < +\infty$

and $\mu_2(\underline{X_i \cap Y_j}) < +\infty$



$X_i \cap Y_j$

We may assume μ_1, μ_2 are totally finite. (7)

If the statement $\mu_1(A) = \mu_2(A)$ $\forall$ $A \in S(\mathcal{C})$ is true when μ_1, μ_2 are totally finite, then it will also be true when μ_1, μ_2 are σ-finite

Let μ_1, μ_2 σ-finite

$$\Longrightarrow \quad X = \bigsqcup_{i=1}^{\infty} X_i, \quad X_i \in \mathcal{C}, \ \mu_1(X_i) < +\infty$$

$$\text{ly} \quad X = \bigsqcup_{j=1}^{\infty} Y_j, \quad Y_j \in \mathcal{C}, \ \mu_2(Y_j) < +\infty$$

Note μ_1, μ_2 restricted to $X_i \cap Y_j$ are totally finite:

$\forall \; A \subseteq X_i \cap Y_j, \quad A \in \mathcal{C}$,

$\mu_1(A) < +\infty$

$\mu_2(A) < +\infty$.

Now $A \subseteq X, \overline{A \in \sigma(\mathcal{C})} \quad A = \biguplus_i \biguplus_j (A \cap X_i \cap Y_j)$

$$\mu_1(A) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \underline{\mu_1(A \cap X_i \cap Y_j)}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_2(A \cap X_i \cap Y_j)$$

$$= \mu_2(A).$$

μ_1, μ_2 totally finite

$\mathcal{C}$ – Algebra

$\mu_1(A) = \mu_2(A) \ \forall A \in \mathcal{C}$

$\Longrightarrow \underline{\mu_1(A) = \mu_2(A)} \ \forall A \in \mathcal{S}(\mathcal{C})$?

Pf $\mathcal{M} := \{E \in \mathcal{S}(\mathcal{C}) \mid \mu_1(E) = \mu_2(E)\}$

<u>Claim</u> $\mathcal{M}$ is a monotone class.

let $\{E_n\}_{n \geq 1}$ in $\mathcal{M}$ s.t $E_n \subseteq E_{n+1} \ \forall n$

To Show $E = \bigcup E_n \in \mathcal{M}$?

<u>Note</u> $E_n \uparrow E = \bigcup_{n=1}^{\infty} E_n$, $\underline{E_n \in \mathcal{M}}$

Note $\emptyset$

$$\mu_1(E) = \lim_{n\to\infty} \mu_1(E_n) \quad (\mu_1 \text{ c.a})$$

$$= \lim_{n\to\infty} \mu_2(E_n) \quad (E_n \in \mathcal{M})$$

$$= \mu_2(E) \quad (\underline{\mu_2 \text{ is c.a}})$$

$$\Rightarrow \quad E \in \mathcal{M}$$

Let $E_n \in \mathcal{M}$, $E_n \supseteq E_{n+1}$ $\forall n$

$$E = \bigcap_{n=1}^{\infty} E_n$$

$$\mu_1(E) = \lim_{n\to\infty} \mu_1(E_n) \quad (\because \mu_1(X) < +\infty, \ \mu_1 \text{ c.a.})$$

$$= \lim_{n\to\infty} \mu_2(E_n) = \mu_2(E)$$

$$\mathcal{M} = \{ E \in S(\mathcal{C}) \mid \mu_1(E) = \mu_2(E) \}$$ (12)

is a Monotone class.

<u>Given</u> $\mu_1(A) = \mu_2(A) \quad \forall A \in \mathcal{C}$

i.e., ✓ $\underline{\mathcal{C} \subseteq \mathcal{M}}$

$\Longrightarrow \quad \mathcal{M}(\mathcal{C}) \subseteq \mathcal{M}$

$\mathcal{C}$ algebra $\Rightarrow \underline{\mathcal{M}(\mathcal{C}) = S(\mathcal{C})}$

$\equiv \quad S(\mathcal{C}) \subseteq \mathcal{M} \subseteq S(\mathcal{C})$

$\Longrightarrow \quad \underline{\mathcal{M} = S(\mathcal{C})}$ ∎

(13)

$\mathcal{C}$ semi-algebra

$\phi, X \in \mathcal{C}$,

$A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$

$A \in \mathcal{C} \Rightarrow A^c = \bigsqcup_{i=1}^{n} C_i$

$C_i \in \mathcal{C}$

$\mathcal{C}$: Algebra

$\phi, X \in \mathcal{C}$

$A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$

$A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

Semi algebra $\Leftarrow$ Algebra

$\not\Rightarrow$

σ-algebra

$\mathcal{C}$

(i) $\phi, X \in \mathcal{C}$

~~$A \cap B \in \mathcal{C}$ if $A, B \in \mathcal{C}$~~

(ii) $A_i \in \mathcal{C} \Rightarrow \bigcap A_i \in \mathcal{C}$

(iii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

$\equiv A_i \in \mathcal{C} \Rightarrow \bigcup A_i \in \mathcal{C}$

Monotone class $\mathcal{M}$

(14)

(i) $E_n \in \mathcal{M}$, $E_n \uparrow$, $E = \bigcup E_n$

$\Longrightarrow E \in \mathcal{M}$ ✓

(ii) $E_n \in \mathcal{M}$, $E_n \downarrow$, $E = \bigcap E_n$

$\Longrightarrow E \in \mathcal{M}$ ✓

σ-algebra $\Longrightarrow$ Monotone class

$\not\Longleftarrow$

$$\mathcal{C} \subseteq P(X)$$

$\underline{\mathcal{A}(\mathcal{C})}$ = Algebra generated by $\mathcal{C}$

$$= \bigcap_{\mathcal{C} \subseteq \mathcal{A}} \mathcal{A}$$

$$\underline{\mathcal{S}(\mathcal{C})} = \bigcap_{\mathcal{C} \subseteq \mathcal{S}} \mathcal{S}$$

$$\mathcal{M}(\mathcal{C}) = \bigcap_{\mathcal{C} \subseteq \mathcal{M}} \mathcal{M}$$

$\boxed{\mathcal{M}(\mathcal{C}) = \mathcal{S}(\mathcal{C}) \text{ if } \mathcal{C} \text{ is an algebra}}$

$$\mu : \mathcal{A} \longrightarrow [0,\infty) \qquad \mu(\phi) = 0$$

algebra

① μ is c.a. $\iff$ μ is c. sub add. + μ σ.a.

② μ σ.a. $\mu(X) < +\infty$

μ c.a. $\iff$ $\begin{cases} E_n \downarrow E \\ \cancel{\Rightarrow \mu(E_n)} \\ \Rightarrow \lim_{n\to\infty} \mu(E_n) = \mu(E) \end{cases}$

$\iff$ $\begin{cases} E_n \uparrow E \\ \Rightarrow \mu(E) = \lim_{n\to\infty} \mu(E_n). \end{cases}$

Uniqueness

$\mu_1, \mu_2 : \Sigma(\mathcal{C}) \longrightarrow [0, +\infty]$ are σ-finite measures

$\mathcal{C}$ is a semi algebra

and $\mu_1(A) = \mu_2(A) \quad \forall A \in \mathcal{C}$

$$\Longrightarrow \quad \mu_1(A) = \mu_2(A) \quad \forall A \in \Sigma(\mathcal{C})$$